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Retracts of strong products of graphs

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Abstract

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Let G and H be connected graphs and let $G*H$ be the strong product of G by H. We show that every retract R of $G*H$ is of the form $R=G'*H'$, where G' is a subgraph of G and H' one of H. For triangle-free graphs G and H both G' and H' are retracts of G and H, respectively. Furthermore, a product of finitely many finite, triangle-free graphs is retract-rigid if and only if all factors are retract-rigid and it is rigid if and only if all factors are rigid and pairwise non-isomorphic.

1. Introduction

The main motivation for this paper is the investigation [9] by Nowakowski and Rival, in which decomposition theorems for retracts of the Cartesian products of graphs are derived for strongly-triangulated and weakly-triangulated graphs as well as for graphs without four-cycles.

The Cartesian product is also considered in [1, 14], where retracts of Hamming-graphs and of hypercubes are characterized. Varieties of graphs with respect to graph retracts and the direct product of graphs were considered in [5, 8, 10, 12, 13]. A connection between n-chromatic absolute retracts and absolute reflexive retracts, using the direct product, is established in [12].

All graphs considered in this paper will be finite or infinite undirected, simple graphs, i.e., graphs without loops or multiple edges. A subgraph R of a graph G

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is a *retract* of G is there is an edge-preserving map $r: V(G) \to V(R)$ with $r(x) = x$, for all $x \in V(R)$. The map r is called a *retraction*. Note that $[x, r(x)] \notin E(G)$. If R is a retract of G then R is an *isometric* subgraph of G, that is $d_G(x, y) = d_R(x, y)$ for all $x, y \in V(R)$, whre $d_H(a, b)$ denotes the distance in H between $a, b \in V(H)$.

The *strong product* $G * H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $[(a, x), (b, y)] \in E(G * H)$ whenever $[a, b] \in E(G)$ and $x = y$, or $a = b$ and $[x, y] \in E(H)$, or $[a, b] \in E(G)$ and $[x, y] \in E(H)$. The strong product is commutative, associative and K_1 is a unit. Also, $G * H$ is connected if and only if both G and H are connected. Whatever possible we shall denote the vertices of one factor by $a, b, c, \ldots$ and the vertices of the other factor by x, y, z.

A *clique* of a graph is a maximal complete subgraph. If K is a clique of the strong product $G * H$ then it is easy to see that $K = G' * H'$, where G' and H' are cliques of G and H, respectively.

If S is a subgraph of $G * H$, then let $p_G(S) = \{a \mid (a, x) \in V(S)\}$. Analogously we define $p_H(S)$.

If $r_1: V(G) \to V(R)$ and $r_2: V(G') \to V(R')$ are retractions, it is easy to see that $r: (a, x) \mapsto (r_1(a), r_2(x))$ is a retraction from $G * G'$ onto $R * R'$. We call this retraction *canonical*. Not every retraction of $G * G'$ is of this form, as can be seen from Fig. 1, where the filled vertices induce a retract and a corresponding retraction is indicated with arrows.

In Section 3 we shall prove that every retraction is indeed canonical when both G and G' are triangle-free and both factors of the retract have at least three vertices.

It is not true that every retract of $G * G'$ is of the form $R * R'$, where R and R' are retracts of G and G', respectively, if G and G' are not triangle-free. A counterexample can be constructed as follows. Denote by H_n, $n \geq 4$, a graph which we get from a copy of the Mycielski graph G_n and the complete graph K_{n-1} by joining an arbitrary vertex of G_n with a vertex of K_{n-1}. One can show that there is a retraction from $V(H_n * K_2)$ onto a subgraph $K_{n-1} * K_2$, but as $\chi(G_n) = n$ there is no retraction $V(G_n) \to V(K_{n-1})$. (For details see [6].)

Nevertheless, we assume that every retract of strong products of a large class of

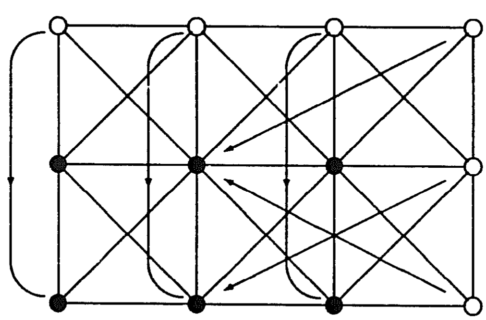

Fig. 1. A retraction.

graphs are products of retracts of the factors. In particular, we conjecture that this is true for products of perfect graphs.

In Section 2 we show that every retract of a strong product is itself a strong product. This contrasts with the case of Cartesian products, in which there are plenty of retracts that are not products of their projections on the factors, see [9].

In the last section we prove that a strong product of finitely many finite, triangle-free graphs is retract-rigid if and only if all factors are retract-rigid and rigid if and only if all factors are rigid and pairwise non-isomorphic.

2. Retracts of strong products are products

Lemma 2.1. *Let R be a retract of $G * H$ and let (a, x), (b, y) be adjacent vertices of R. Then $(a, y) \in V(R)$ and $(b, x) \in V(R)$.*

Proof. Nothing has to be proved if $a = b$ or $x = y$. Suppose therefore that $a \neq b$ and $x \neq y$. Clearly, the set of vertices $\{(a, x), (b, x), (a, y), (b, y)\}$ induces a complete graph in $G * H$. Due to symmetry it is enough to prove that $(b, x) \in V(R)$. Let $r : V(G * H) \to V(R)$ be a retraction and suppose $(b, x) \notin V(R)$. Set $(c, z) = r(b, x)$. Clearly, $(c, z) \neq (b, x)$. Since (c, z) is adjacent to (b, y) we have $b = c$ or $[b, c] \in E(G)$. Similarly, since (c, z) is adjacent to (a, x), we infer $x = z$ or $[x, z] \in E(H)$. Thus $(c, z) = r(b, x)$ is adjacent to (b, x), a contradiction. □

Theorem 2.2. *Let G and H be connected grahs and let R be a retract of $G * H$. Then $R = G' * H'$, where G' and H' are subgraphs of G and H.*

Proof. It suffices to show that $(a, x), (b, y) \in V(R)$ implies that $(a, y) \in V(R)$ and $(b, x) \in V(R)$. We may suppose that the vertices (a, x), (b, x), (a, y) and (b, y) are pairwise different. Let P be a shortest $(a, x) - (b, y)$ path in R and let $|P| = n$. Such paths always exist since R is an isometric subgraph.

We claim that $(a', x') \in V(R)$ for all $a' \in p_G(P)$, $x' \in p_H(P)$.

If $n = 1$, the claim is precisely Lemma 2.1. Suppose now that the claim is true for every pair of vertices in R of distance less than n. Let (a_1, x_1) by the first vertex on P different from (a, x) and let P' denote the $(a_1, x_1) - (b, y)$ subpath of P. By the induction hypothesis, $(a', x') \in V(R)$ for $a' \in p_G(P')$, $x' \in p_H(P')$. Since $p_G(P) = p_G(P') \cup \{a\}$ and $p_H(P) = p_H(P') \cup \{x\}$ we must show that $(a, x') \in V(R)$ for all $x' \in p_H(P)$, and $(a', x) \in V(R)$ for all $a' \in p_G(P)$.

Let (a, x') be an arbitrary vertex with $x' \in p_H(P)$. As $p_H(P) = p_H(P') \cup \{x\}$ we may assume that $x' \in p_H(P')$. Then $p_H(P')$ contains a $y - x'$ path, say P'', and, by the induction hypothesis, all the vertices of $\{a_1\} * P''$ are in $V(R)$. If $a_1 = a$ then immediately $(a, x') \in V(R)$.

Otherwise, by Lemma 2.1, $(a, x_1) \in V(R)$. By the induction hypothesis again, it is clear that every vertex of $\{a\} * P''$ belongs to $V(R)$ and hence $(a, x') \in V(R)$ for $x' \in p_H(P)$. Interchanging the roles of G and H we see that all vertices (a', x), $a' \in p_G(P)$, also belong to $V(R)$. □

We further wish to show that G' and H' are isometric subgraphs of G and H, respectively. It suffices to prove this for the first factor. In order to do this we first observe that the layers $\{a\} * H$ and $G * \{x\}$ of any product $G * H$ are isometric subgraphs of this product. Hence, $G' * \{x\}$ is isometric in $G' * H'$, and since $G' * H'$ (as a retract of $G * H$) is isometric in $G * H$, the layer $G' * \{x\}$ is also isometric in $G * H$. But then it must be isometric in any subgraph of $G * H$ containing it, in particular in $G * \{x\}$, which in turn implies that G' is isometric in G.

In a private communication Bandelt observed that the graphs G' and H' of Theorem 2.2 are so-called reflexive retracts of G and H, respectively. (A reflexive retract is a retract in which edges can be mapped into single vertices.) This observation gives an alternate proof that G' and H' are isometric subgraphs, since reflexive retracts are also isometrically embedded.

We further note that a generalization of the isometric subgraph condition to 'holes' (also called 'gaps' in [8]) was introduced by Nowakowski and Rival in [8], see also [5]. A *hole* of a graph G is a pair (K, δ), where K is a non-empty set of vertices of G and δ a function from K to nonnegative integers such that no $x \in V(G)$ has $d_G(x, y) \leq \delta(y)$ for all $y \in K$. An *m-hole* is a hole (K, δ) with $|K| = m$. A hole (K, δ) of a subgraph H of G is *separated* in G if (K, δ) is also a hole of G. Being an isometric subgraph is equivalent to having all 2-holes separated.

It follows from Bandelt's observation that every hole of G' and H' is separated in G, resp. H, which supports our conjecture that a retract of the strong product $G * H$ of a large class of graphs is the strong product of retracts of G and H.

3. Triangle-free graphs

Theorem 3.1. *Let G be a connected, triangle-free graph and H a connected graph. Let R be a retract of $G * H$ and $r: V(G * H) \to V(R)$ a retraction. Then $R = G' * H'$ and there is a retraction r_G from G onto G' with $r(\{a\} * Q) = \{r_G(a)\} * Q$ for any clique Q of H' and any $a \in V(G)$.*

Proof. Let G be a connected, triangle-free graph and H a connected graph. We may suppose that both G and H are nontrivial. Let $r: V(G * H) \to V(R)$ be a retraction. By Theorem 2.2, $R = G' * H'$, and by the above H' is isometric in H.

We first wish to show that both G' and H' are nontrivial. To see this, we note that the cliques of $G * H$ are the strong product of the cliques of G and H, respectively and consider maximum cliques C_1 and G and C_2 of H. Clearly $C_1 * C_2$ is a maximum clique of $G * H$ and hence also $r(C_1 * C_2)$, because retractions do not identify adjacent vertices. Therefore, $G' * H'$ has to contain the product of a maximum clique of G by a maximum clique of H, and thus, both G' and H' have to contain at least two vertices each.

The proof of the theorem is by induction on the distance of the vertex $a \in V(G)$ from G'. Let Q be a clique of H' and let $a \in V(G) - V(G')$ and $b \in V(G')$ be adjacent vertices. Let $x \in Q$ and $r(a, x) = (a', x')$. Since (a, x) is adjacent to all vertices of $\{b\} * Q$ we clearly have $a' \neq b$. Furthermore, $x' \in Q$, for otherwise the vertices $Q \cup \{x'\}$ would induce a complete graph in H', which properly contains the clique Q. Let y be any vertex from Q, $y \neq x$ and let $r(a, y) = (b', y')$. Clearly $b' \neq b$. If $b' \neq a'$ then the vertices $\{a', b, b'\}$ induce a triangle in G'. Hence, for any $y \in Q$, we have $r(a, y) = (a', y')$, where $y' \in Q$. Since Q is complete, $r(\{a\} * Q) = \{a'\} * Q$. If Q' is any other clique of H' then $r(\{a\} * Q') = \{a''\} * Q'$. Hence $a' = a''$, if $Q' \cap Q \neq \emptyset$, and therefore $r(\{a\} * H') = \{a'\} * H'$.

Suppose now that for every vertex a of distance less than k, $k \geqslant 2$, from G', $r(\{a\} * Q) = \{a'\} * Q$, for some $a' \in V(G)$. Let $d(b, G') = k$ and $d(c, G') = k - 1$, where c is adjacent to b. Choose $x, y \in Q$ and let $r(b, x) = (b', x')$ and $r(b, y) = (b'', x'')$. Furthermore, let $c' \in V(G')$ be a vertex with $r(\{c\} * Q) = \{c'\} * Q$. Clearly, $x', x'' \in Q$, for otherwise $Q \cup \{x'\}$ or $Q \cup \{x''\}$ would induce a complete graph in H', which properly contains Q. We also note that $c' \neq b', b''$. If $b' \neq b''$ then the vertices $\{b', b'', c'\}$ induce a triangle of G'. Hence, $r(\{b\} * Q) = \{b'\} * Q$ for some $b' \in V(G')$. Define r_G by setting $r_G(b) = b'$. It easily follows that r_G is a retraction of G onto G'. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1, $r(\{a\} * H') = \{r_G(a)\} * H'$ *for any* $a \in V(G)$.

Corollary 3.3. *Let* $G = K_n * H$, *where* H *is connected and triangle-free. Then every retract* R *of* G *is of the form* $K_n * H'$, *where* H' *is a retract of* H.

Theorem 3.4. *Let* G *and* H *be connected, triangle-free graphs. Then* R *is a retract of* $G * H$ *if and only if* $R = G' * H'$, *where* G' *is a retract of* G *and* H' *is a retract of* H. *Furthermore, if* $|V(G')| \geqslant 3$, $|V(H')| \geqslant 3$ *and if* $r : V(G * H) \to V(R)$ *is a retraction, then* r *is canonical.*

Proof. Let G and H be connected, nontrivial, triangle-free graphs and let $r : V(G * H) \to V(R)$ be a retraction. By Theorem 2.2, $R = G' * H'$, and by Theorem 3.1 G' is a retract of G and H' is a retract of H.

Assume next $|V(G')| \geqslant 3$ and $|V(H')| \geqslant 3$. Let $x \in V(H')$ be any vertex with degree at least two in H' and let $y, z \in V(H')$ be adjacent to x. Let a be any vertex in $V(G) - V(G')$. Observe that the cliques of H, and in particular of H', are isomorphic to K_2. Hence according to Theorem 3.1 $r(a, x) \in \{(b, x), (b, y)\}$

and $r(a,x) \in \{(b',x),(b',z)\}$, for some $b, b' \in V(G')$. It follows $b'=b$ and $r(a,x)=(b,x)$. This implies also that $r(a,y)=(b,y)$ and $r(a,z)=(b,z)$. Therefore, by induction with respect to the distance between x and x', we have $r(a,x')=(b,x')$ for all $x' \in H'$. Set $b=r_G(a)$. By symmetry, if x is any vertex from $V(H)-V(H')$ then $r(a',x)=(a',y)$ for all $a' \in V(G')$, where $y \in V(H')$. Set $y=r_H(x)$. Finally, for $a' \in V(G')$ and $x' \in V(H')$ set $a'=r_G(a')$ and $x'=r_H(x')$, respectively.

We wish to show that $r(a,x)=(r_G(a), r_H(x))$. By the above we only have to consider the case when $a \in V(G)-V(G')$ and $x \in V(H)-V(H')$.

Let (a,x) be a vertex with $a \in V(G)-V(G')$, $x \in V(H)-V(H')$ and $d(a,G')=1$, $d(x,H')=1$. Let $a' \in V(G')$ be a vertex adjacent to a and let $x' \in V(H')$ be a vertex adjacent to x. Now $r(a',x)=(a', r_H(x))$ and $r(a,x')=(r_G(a),x')$. It follows that $r(a,x)=(r_G(a),r_H(x))$, for otherwise $\{a', r_G(a'), r_G(a)\}$ would induce a triangle in G' or $\{x', r_H(x'), r_H(x)\}$ would induce a triangle in H'.

Again we can show by induction that $r(b,x)=(r_G(b),r_H(x))$ for every $b \in V(G)-V(G')$, and then that $r(b,y)=(r_G(b),r_H(y))$ for every $b \in V(G)-V(G')$ and $y \in V(H)-V(H')$. □

To show that the restriction on the number of vertices of G and H in the second part of the Theorem cannot be relaxed, we refer again to Fig. 1, which shows a noncanonical retract with $|V(G')|=2$ and $|V(H')|=3$.

4. Rigid strong products of graphs

A graph G is *asymmetric* if its automorphism group $\mathrm{Aut}(G)$ is trivial. G is called *rigid* if it has no proper endomorphism and *retract-rigid* if it has no proper retraction. We wish to characterize rigid and retract-rigid strong products of graphs. Hell [3, Proposition 6] established an important connection between rigid and retract-rigid graphs.

Theorem 4.1. *A finite graph is rigid if and only if it is asymmetric and retract-rigid.*

It would help our investigations if the automorphism group of the strong product were the product of the automorphism groups of its factors. This is almost the case if a certain relation on $G * H$ is trivial. To make this more precise, we introduce an equivalence relation $S(G)$ on the vertex set $V(G)$ of a graph G, defined as follows: $x\,S(G)\,y$ whenever:

(i) $[x,y] \in E(G)$ or $x=y$, and

(ii) every $z \in V(G)$, $z \neq x, y$, is either adjacent to both x and y or to neither of them.

Furthermore we define a graph G/S by $V(G/S) = V(G)/S$ and by connecting two vertices $X, Y \in V(G/S)$ if and only if there exist $x \in X$ and $y \in Y$ such that $[x, y] \in E(G)$. We then have (see [2, Lemma 4]) the following.

Lemma 4.2. *Let G be a finite graph and let G be the strong product $\prod_{i=1}^{n} G_i$. Then $G/S = \prod_{i=1}^{n} G_i/S$.*

For the description of $\mathrm{Aut}(G/S)$ we introduce two more definitions. Let $G = G_1 * G_2$ and let α_1 and α_2 be automorphisms of G_1 and G_2, respectively. Then $\beta(a, x) = (\alpha_1 a, \alpha_2 x)$ is an automorphism of G. It is called the *direct product* of the automorphisms α_1 and α_2. If $\alpha : G_1 \to G_2$ is an isomorphism, then $\beta(a, x) = (\alpha a, \alpha^{-1} x)$ is an automorphism of G. It is called an *interchange* of the factors G_1 and G_2. Now we have (see [2, Satz 9]) the following.

Theorem 4.3. *Let G be the strong product $\prod_{i=1}^{n} G_i$, where the G_i are finite, indecomposable, connected graphs. Then* $\mathrm{Aut}(G)$ *is generated by the direct products of the automorphisms of the G_i and by interchanges of the G_i if and only if $G/S \cong G$.*

Note that triangle-free graphs are indecomposable. We also observe, that the decomposition of a finite, connected graph into the strong product of indecomposable factors is unique, i.e., the prime factorization with respect to the strong product is unique. This was shown by Mc Kenzie [7], and independently by Dörfler and Imrich [2].

We shall use the following corollary to Theorem 4.3.

Corollary 4.4. *Let G be the strong product $\prod_{i=1}^{n} G_i$, where the graphs G_i are connected and triangle-free, and let R be a retract of G. Then $R = \prod_{i=1}^{n} R_i$, where each R_i is a retract of G_i, $1 \leq i \leq n$.*

Proposition 4.5. *Let G be the strong product $\prod_{i=1}^{n} G_i$, where the G_i are connected and triangle-free. Then G is retract-rigid if and only if all the G_i are retract-rigid.*

Proof. If R_j is a proper retract of G_j then $R_j * \prod_{i \neq j} G_i$ is a proper retract of G.

If R is a proper retract of G then by Corollary 4.4, $R = \prod_{i=1}^{n} R_i$, where the R_i are retracts of the G_i. Hence, there exists a factor G_j, such that R_j is a proper retract of it. □

Theorem 4.6. *Let G be the strong product $\prod_{i=1}^{n} G_i$, where the G_i are finite, connected, triangle-free nontrivial graphs. Then G is rigid if and only if the G_i are pairwise non-isomorphic and rigid.*

Proof. Since each G_i is a finite, connected and triangle-free graph, $G_i/S \cong G_i$. Hence by Lemma 4.2, $G/S \cong G$.

Let G be a rigid graph. Then the G_i must be pairwise non-isomorphic, for otherwise the interchanges of factors would determine nontrivial automorphisms of G. Since G is rigid, it is retract-rigid and hence, by Propositon 4.5, all the G_i are retract-rigid. As $G/S = G$ and G is asymmetric, it follows from Theorem 4.3 that all the G_i are asymmetric. Hence, by Theorem 4.1, all the G_i are rigid.

Conversely, let G_i be pairwise non-isomorphic and rigid. Since the G_i are asymmetric and since there are no interchanges of factors, it follows from Theorem 4.3 that G is asymmetric. By Proposition 4.5, G is also retract-rigid and therefore rigid. □

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